Rational-driver approximation in car-following theory

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The problem of a car following a lead car driven with constant velocity is considered. To derive the governing equations for the following car dynamics a cost functional is constructed. This functional ranks the outcomes of different driving strategies, which applies to fairly general properties of the driver behavior. Assuming rational-driver behavior, the existence of the Nash equilibrium is proved. Rational driving is defined by supposing that a driver corrects continuously the car motion to follow the optimal path minimizing the cost functional. The corresponding car-following dynamics is described quite generally by a boundary value problem based on the obtained extremal equations. Linearization of these equations around the stationary state results in a generalization of the widely used optimal velocity model. Under certain conditions (the "dense traffic" limit) the rational car dynamics comprises two stages, fast and slow. During the fast stage a driver eliminates the velocity difference between the cars, the subsequent slow stage optimizes the headway. In the dense traffic limit an effective Hamiltonian description is constructed. This allows a more detailed nonlinear analysis. Finally, the differences between rational and bounded rational driver behavior are discussed. The latter, in particular, justifies some basic assumptions used recently by the authors to construct a car-following model lying beyond the frameworks of rationality.

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I. CAR-FOLLOWING THEORIES AND BASIC PROPERTIES OF DRIVER BEHAVIOR

Recently, the theoretical and empirical foundations of the physics of traffic flow (for a review, see Refs. [1,2]) have come into the focus of the physical community. The motion of individual cars has many peculiarities, since it is controlled by motivated driver behavior, together with some physical boundaries. Nevertheless, on macroscopic scales the vehicle ensembles display phenomena such as phase formation and phase transitions widely met in physical systems (see, e.g., Refs. [1–3]). So, the cooperative behavior of cars treated as active particles seems to be of a more general nature than the mechanical laws and constructing a consistent theory of traffic flow "from scratch" is up to now a challenging problem.

To describe individual car dynamics a great variety of microscopic models have been proposed. These models differ in the details of the interaction between cars and the time update rule, ranging from differential equations to cellular automata [1,2]. There has been a big deal of work on the macroscopic behavior emerging from the microscopic dynamics when exploring the behavior of systems of interacting cars. However, there is still a lot of controversy in both the macroscopic behavior when compared to reality [4], and in the microscopic foundations of the individual car dynamics itself [5].

One currently adopted approach to specify the microscopic governing equations of the individual car motion is the so-called social force model. More details can be found in Refs. [6–8]; here only the basic ideas are touched on. At each moment t of time, a driver α changes the speed v_α of her car depending on the road conditions and the arrangement of the neighboring cars:

$$\frac{dv_\alpha}{dt}=g_\alpha(v_\alpha)+\sum_{\alpha'\neq\alpha} g_{\alpha\alpha'}(x_\alpha,v_\alpha|x_{\alpha'},v_{\alpha'}). \tag{1.1}$$

The term $g_\alpha(v_\alpha)$ describes the motion of car α on the empty road, whereas the term $g_{\alpha\alpha'}(x_\alpha,v_\alpha|x_{\alpha'},v_{\alpha'})$ allows for the interaction of car α with car α' ($\alpha'\neq\alpha$). The interaction is due to the necessity for driver α to keep a certain safe headway distance between the cars. All the models mentioned above use various *Ansätze* for the last term.

The most interesting special case, which covers the majority of all traffic flow situations, is that of single-lane traffic. Here, all cars can be ordered according to their position on the road in the car motion direction $x_\alpha<x_{\alpha+1}$, here $\alpha=1,\dots,N$. Most models take into account solely nearest neighboring cars α and $\alpha+1$, i.e., $g_{\alpha\alpha'}\neq0$ for $\alpha'=\alpha+1$ and, may be, $\alpha'=\alpha-1$ only. However, more complicated models exist that can be described as models with anticipation [9–13] or the so-called intelligent driver model [14,15].

The earliest "follow-the-leader" models [16,17] relate the acceleration a_α of car α to the velocity difference $(v_\alpha-v_{\alpha+1})$ only, i.e.,

$$\frac{dv_\alpha}{dt}=-\frac{1}{\tau_v}(v_\alpha-v_{\alpha+1}), \tag{1.2}$$

where τ_v is the characteristic time scale of the velocity relaxation. In subsequent generalizations of this model τ_v became a function of the car motion state, in particular, of the current velocity v_α and the headway $h_\alpha=x_{\alpha+1}-x_\alpha-\ell$ (for a review, see Refs. [5,18]). Here, ℓ is the car length. In Refs. [19,20] another approach called the optimal velocity model is proposed, which describes the individual car motion by

$$\frac{dv_\alpha}{dt} = -\frac{1}{\tau_v}[v_\alpha - \vartheta_{\mathrm{opt}}(h_\alpha)], \tag{1.3}$$

where $\vartheta_{\mathrm{opt}}(h)$ is the steady-state velocity (the optimal velocity) chosen by drivers as function of the headway h between the cars. It should be noted that this approach is related to much earlier safety distance models [21–23].

Concerning the fundamentals of approximations such as $a_\alpha = a(v_\alpha, v_{\alpha+1}, x_\alpha, x_{\alpha+1})$, it is noted that there are actually two stimuli affecting the driver behavior. One of them is the necessity to move at the mean speed of traffic flow, in the given case at the speed $v_{\alpha+1}$ of car $\alpha+1$. So, first, driver α should control the velocity difference $(v_\alpha - v_{\alpha+1})$. The other is the necessity to maintain a safe headway distance $h_{\mathrm{opt}}(v_\alpha)$ depending on the current velocity v_α. The following-the-leader models mainly take into account the former stimulus. The optimal velocity model, conversely, allows for the latter stimulus only. More sophisticated approximations, e.g., Refs. [14,15,24–28] to name but a few, allow for both stimuli. Note also a simple *Ansatz* called the combined model in Ref. [29] which is also related to the intelligent driver model [14,15]:

$$\frac{dv_\alpha}{dt} = -\frac{(1-\kappa)}{\tau_v}[v_\alpha - v_{\alpha+1}] - \frac{\kappa}{\tau_v}[v_\alpha - \vartheta_{\mathrm{opt}}(h_\alpha)]. \tag{1.4a}$$

This equation takes into account both stimuli via a phenomenological coefficient $0<\kappa<1$. This is also the case for the Helly model [24] which can be written as (cf. Ref. [5])

$$\frac{dv_\alpha}{dt} = -\frac{1}{\tau_v}[v_\alpha - v_{\alpha+1}] + \frac{1}{\tau_v L_H}[(x_{\alpha+1} - x_\alpha) - h_{\mathrm{opt}}(v_\alpha)], \tag{1.4b}$$

where L_H is a certain spatial scale and $h_{\mathrm{opt}}(v)$ is the optimal headway distance chosen by drivers when moving at speed v. Reference [24] used a linear *Ansatz* for $h_{\mathrm{opt}}(v)$. Later [26], this model was generalized to allow for the dependence of the kinetic coefficients on the motion state.

However, the question whether a collection of variables such as $\{v_\alpha, v_{\alpha+1}, x_\alpha, x_{\alpha+1}\}$ specifies the acceleration a_α completely is not trivial. Drivers are characterized by the motivated behavior rather than physical regularities. For example, memory effects may be essential and can destroy the direct relationship $a_\alpha = a(v_\alpha, v_{\alpha+1}, x_\alpha, x_{\alpha+1})$. Up to now, memory effects in car-following modeling have been treated only in a simplified version. This has been done by relating the current acceleration $a_\alpha(t)$ to the velocities $v_\alpha(t-\tau_a)$, $v_{\alpha+1}(t-\tau_a)$ and the headway distance $h_\alpha(t-\tau_a)$ taken at a previous moment $(t-\tau_a)$ of time (for a review of such an approach concerning with the following-the-leader models, see Refs. [5,18], regarding the optimal velocity model see Refs. [30–32]). Here τ_a is the formal delay time in the driver response which is treated as a constant. Such an approach, however, is rather formal, because, first, it is not clear why the memory effects relate only two moments of time instead of a certain interval as a whole. Second, a simple physiological delay in the driver response as well as the mental driver estimation of the surrounding situation should contribute to the value of τ_a. If the latter contribution is essential the delay time τ_a is to depend substantially on the state of car motion. Moreover, from our point of view the memory effects stem from the fact that the description of individual car motion is a boundary value problem rather than an initial value problem. Indeed, at the current moment of time the headway distance and the car velocity are quantities given for the driver beforehand. Correcting the car motion she should choose such a driving strategy that in a certain time interval the car velocity and headway distance attain their optimal stationary values, at least, approximately.

These memory effects are the topic of this paper. We propose that drivers plan their behavior for a certain time in advance [33] instead of simply reacting to the surrounding situation. Similar ideas related to the optimum design of a distance controlling driver assistance system are discussed in Ref. [34]. Mathematically, the driver's planning of the further motion is just to find extremals of a certain priority functional that ranks outcomes of different driving strategies.

The derivation of microscopic governing equations for systems with motivated behavior based on a certain "optimal self-organization" principle has been discussed recently [2,35–39]. The assumption adopted in these works is that individuals try to minimize the interaction strength or, equivalently, to optimize their own success and to minimize the efforts required for this. The approach discussed here applies to the concepts of mathematical economics, namely, to the notion of preferences and utility (see, e.g., Ref. [40]). In Ref. [33] a specific form of the priority functional has been proposed. From that, a certain *Ansatz* such as the combined model (1.4a) or the Helly model (1.4b) could be *derived*. Nevertheless, the question how to find the priority functional "from scratch" remains open.

In the present paper the priority functional is constructed by applying to general properties of the driver behavior for a simplified situation. A car following (no overtaking allowed) a lead car which is driven with constant speed V is considered. The task is to derive governing equations for the following car motion specified by the time dependence of the velocity $v(t)$ and the headway distance $h(t)$.

II. THE COST FUNCTION OF DRIVING

A. General properties of driver preference

Assuming a driver to be able of comparing any two states $\{h_1, v_1\}$, $\{h_2, v_2\}$, the phase plane $\{h>0, v>0\}$ can be ordered by a preference relation $\preccurlyeq$. Therefore, the existence of a cost function $\mathcal{F}(h,v)$ may be assumed such that

$$\{h_1, v_1\} \preccurlyeq \{h_2, v_2\} \Leftrightarrow \mathcal{F}(h_1, v_1) \geqslant \mathcal{F}(h_2, v_2). \tag{2.1}$$

Typically, $-\mathcal{F}(h,v)$ is called the utility function and (for it) the relation $\preccurlyeq$ matches the inequality $\leqslant$. Here, the use of condition (2.1) is preferred, because then the cost function $\mathcal{F}(h,v)$ is similar to the free energy of physical systems, with the minima as the stationary states.

Obviously, the cost function $\mathcal{F}(h,v)$ cannot be specified completely because a composite function $\Psi[\mathcal{F}(h,v)]$ also meets condition (2.1) for any increasing function $\Psi[\cdot]$. Therefore, at the current stage of the theory development any approximation or *Ansatz* adopted for the cost function $\mathcal{F}(h,v)$ has no meaning.

Applying to a simple speculation [41] it can be shown that if a path goes through a certain small neighborhood of the point $\{h,v\}$ one or many times then only the cumulative time interval δt during which the system was located in the given neighborhood contributes to the cost functional. The corresponding weight coefficient $\mathcal{F}(h,v)$ obviously plays the role of a certain cost function for the given traffic state. So, the resulting cost functional for the whole path is

$$\mathcal{L}\{h(t),v(t)\}=\int_{\text{origin}}^{\text{destination}}\mathcal{F}(h(t),v(t))dt. \tag{2.2}$$

If the driver has incomplete information about the traffic flow pattern ahead she plans the further motion only within some limits, spatial and temporal ones. This can be taken into account by introducing a certain cofactor in front of the function $\mathcal{F}(h,v)$ which depends on spatial coordinates or time.

The cost function $\mathcal{F}(h,v)$ entering the cost functional (2.2) is already determined within a constant multiplier and a certain additive term not affecting driver's preference [41]. This feature enables us to seek for a reasonable *Ansatz* for the given cost function.

B. Characteristic features of the car motion cost

In the following, the priority function of the car motion state with respect to the velocity v and the reciprocal value $\rho=1/h$ of the headway distance will be characterized. Note that the value ρ is not the real car density on a highway. The present analysis ignores the car length, so we have preferred to use the value ρ defined as above. The most preferable state is the motion on an empty road ($\rho=0$) at maximum speed $\vartheta_{\max}$. This maximum speed is determined by external factors. So, at $\{\rho=0,v=\vartheta_{\max}\}$ the cost function $\mathcal{F}(\rho,v):=\mathcal{F}(h,v)|_{h=1/\rho}$ has its global minimum:

$$\mathcal{F}(0,\vartheta_{\max})=0, \tag{2.3}$$

set equal to zero keeping in mind the aforesaid about the freedom in specifying the cost function. Since there is no other minimum for the motion on an empty road,

$$\mathcal{F}(0,0)=1, \tag{2.4}$$

can be fixed.

Because of the car construction a driver can visually control the headway distance within some value $l\sim 1$–2 m. When the headway distance h attains such values a driver has to stop her car because of possible collision even at sufficiently slow velocities. So, l coincides with the characteristic headway distance in dense jams where the car density attains the possible maximum. In the following, all headway distances are related to the scale l and the dimensionless variable ρl is used.

When not moving at all ($v=0$) and the density of cars surrounding the given car does not come close to the limit values, i.e., $\rho l\ll 1$, it does not matter how many cars are located in the vicinity. So the assumption that at $v=0$ and for $h\gg l$ the cost function is independent of ρ can be used:

$$\mathcal{F}(\rho,0)=1 \quad \text{for} \quad \rho l\ll 1. \tag{2.5}$$

In all other cases, $\mathcal{F}(\rho,v)$ depends on certain combinations of ρ and v rather than on ρ individually, at least when $\rho l\ll 1$.

Considering the behavior of $\mathcal{F}(\rho,v)$ for a fixed speed v it can be stated that driving with small values of h (large values of ρ) requires a lot of effort. Thus, $\mathcal{F}(\rho,v)$ increases with ρ. The opposite case of small values of ρ (large values of h) deserves special attention. Without the possibility of overtaking no especially attractive headway distance can be marked. Therefore, we assume that the cost function $\mathcal{F}(\rho,v)$ possesses the only one minimum attained at the boundary point $\rho=0$ provided the velocity v is fixed.

Keeping in mind the aforesaid it is reasonable to write

$$\mathcal{F}(\rho,v)=\mathcal{F}(0,v)+f\left(\frac{v}{\vartheta_{\max}}\right)(\rho l)^m \tag{2.6}$$

for $\rho l\ll 1$. Here the exponent m is a constant and the function $f(z)\to 0$ as $z\to 0$. Since the cost function attains its minimum at the boundary point we may set $m=1$. Moreover, the latter term on the right-hand side of expression (2.6) can be interpreted as a certain "interaction" potential between the following and lead cars which is long-distant one for $m=1$. A detailed analysis of effects caused by the value of the exponent m requires an individual study. Here the value $m=1$ is actually chosen as just a simple reasonable assumption. Since the effect of the surrounding cars is depressed for small values of the car velocity v the function $f(z)$ is also to attain its minimum at $z=0$. Therefore, $f(z)=z^2$ as $z\ll 1$ should hold. In other words, inside a certain neighborhood of the origin $\{\rho=0,v=0\}$ the expansion

$$\mathcal{F}(\rho,v)=\mathcal{F}(0,v)+\frac{v^2}{\vartheta_{\max}^2}\rho l \tag{2.7}$$

can be adopted.

Taking into account these speculations about the behavior of the cost function $\mathcal{F}(\rho,v)$ caused by variations of both its arguments, the following simple *Ansatz* will be used subsequently:

$$\mathcal{F}(\rho,v)=\left(1-\frac{v}{\vartheta_{\max}}\right)^2+\frac{v^2}{\vartheta_{\max}^2}\rho l. \tag{2.8}$$

Figure 1 displays this function. It has only one global minimum at $\rho=0$ and $v=\vartheta_{\max}$. For a fixed velocity v it attains a local minimum at the boundary $\rho=0$. *Ansatz* (2.8) generalizes the adopted assumptions about the cost function. The former term takes into account relations (2.3) and (2.4), the latter one is based on approximation (2.7). Of course, the

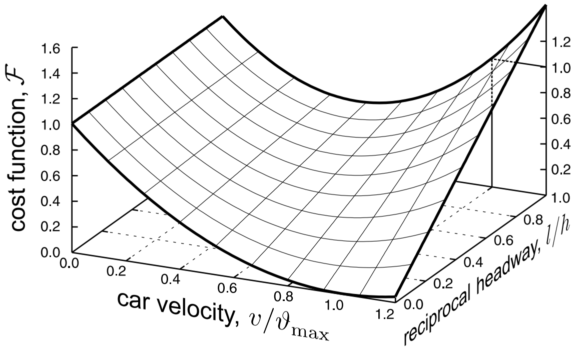

FIG. 1. Characteristic form of the cost function, Eq. (2.8).

parabolic *Ansatz* (2.8) is approximate only in the limit $\rho l \ll 1$. Since this is the main region of interest, this does not constrain its usefulness.

To deal with the car dynamics we should construct the cost functional $\mathcal{L}\{h(t)\}$ for the car motion paths $\{h(t)\}$ treated now as continuous functions of time t. We note that the time dependence of headway distance $h(t)$ gives us the complete information about the car dynamics due to the relationship $dh/dt = V - v$. Leaping ahead, we say that facing this problem it is necessary to introduce additional notions. First, we should expand the phase space in describing the car motion state because transient processes are now the subject of consideration. Second, drivers plan their behavior for a certain time in advance instead of simply reacting to the surrounding situation. So we should specify the region inside which a driver can monitor the traffic flow evolution and, thus, plan driving her car. A priority functional similar to Eq. (2.2) must span the time interval corresponding to this region. Beyond it the contribution of the path fragments to evaluating the car motion quality at the given moment of time has to be fairly minor.

III. RATIONAL DYNAMICS OF CAR MOTION

A. Cost functional and the extremal equation

Dealing with transient processes in the car motion we should consider once more the collection of phase variables characterizing the cost of car motion at the current moment of time. Keeping in mind conventional driver experience, we will expand the current state of car motion given by headway h and velocity v with the car acceleration a. This is essential because a driver cannot change the position and velocity of her car immediately; they vary continuously in time and contain no sharp jumps. Conversely, a driver controls the acceleration directly governing the car motion. Besides, she can change the acceleration practically without delay because in the present analysis it is quite reasonable to ignore time scales related to physiological properties of the driver or to the mechanical properties of the car. So, the cost function $\mathcal{F}^d(h,v,a)$ for the motion state $\{h,v,a\}$ can be written:

$$\mathcal{F}^d(h,v,a) = \mathcal{F}(h,v) + \frac{\tau^2 a^2}{\vartheta_{\max}^2}, \tag{3.1}$$

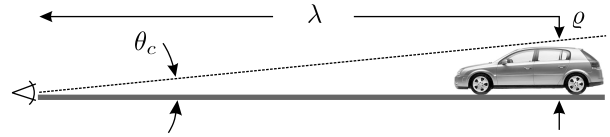

FIG. 2. Illustration of the recognition distance λ and its relationship with the limit angle of perception θ_c.

where the time scale $\tau \gtrsim 1$ s characterizes the acceleration capability of the car. In writing this expression we have assumed that driving without acceleration is preferable. Then, the cost function $\mathcal{F}_d(h,v,a)$ had been expanded into a Taylor series with respect to a, keeping the leading term only. According to the result to be obtained the time scale τ entering expression (3.1) and the one used in Eq. (1.4a) are practically the same. It should be noted that confining ourselves to expansion (3.1) with respect to a the difference between acceleration and deceleration processes has been lost. In reality, they are different. Ignoring this difference leads to models where cars can crash. It is possible to take into account this effect using the approach under development, which, however, is worthy of individual investigation and will be done somewhere else.

For a real driver, various thresholds in the driver recognition of hazards and obstacles exist [42]. One of them is the distance λ at which a driver can recognize the behavior of other objects. This distance is usually related to the threshold of the visual angle θ_c subtended, for example, by vehicles ahead (Fig. 2). The value of θ_c can be estimated as $\theta_c \sim 15$–30 min of arc [43]. The critical angle θ_c, the characteristic height ϱ of cars, and the corresponding mean distance λ are related by

$$\theta_c \sim \frac{\varrho}{\lambda}.$$

This allows for the estimation of the recognition distance as $\lambda \sim 200$–400 m setting $\varrho \sim 2$ m and $\theta_c \sim 30$–15 min of arc. As previously [33], we relate the driver anticipation with the region of size λ in front that is clearly observable and where she can recognize the car behavior. A driver plans her motion based on the information received by monitoring traffic flow inside the observable region. Under normal conditions this region should enable her to govern the motion effectively, for example, to decelerate in advance avoiding a possible accident. Therefore its size λ has to meet the inequality $\lambda \gtrsim \vartheta_{\max}\tau$. Leaping ahead, we introduce the value

$$\sigma = \frac{\vartheta_{\max}\tau}{\lambda} \ll 1 \tag{3.2}$$

treated as a small parameter in the theory to be developed. In particular, for $\tau \approx 1$ s, $\vartheta_{\max} \approx 100$ km/h, and $\lambda \approx 300$ m we have $\sigma \approx 0.1$.

The size λ of the recognition distance can be estimated using another argumentation. If $\tau_d \sim 10$ s is the typical deceleration time from the velocity $\vartheta_{\max}$ attained on the given empty road to zero value then the estimate $\lambda \sim \vartheta_{\max}\tau_d$ should hold. For $\vartheta_{\max} \sim 100$ km/h we have again $\lambda \approx 300$ m.

Now the cost functional of car motion can be written in an integral form as expression (2.2) containing integrand (3.1). The car is assumed to be located at point x along the road and to move with speed v at the current moment t of time. The possible paths of further motion $\{\mathfrak{x}(\mathfrak{t},t),\mathfrak{t}\geqslant t\}$ form the set $\mathfrak{S}(t,x,v)$ on which the cost functional is defined. Here and below Gothic letters will be used to label the path variables. The function $\{\mathfrak{x}(\mathfrak{t},t)\}$ allows for a derivation of all the dynamical variables, the headway distance $\mathfrak{h}(\mathfrak{t})$, the car velocity $\mathfrak{u}(\mathfrak{t})$, and the acceleration $\mathfrak{a}(\mathfrak{t})$ of the trial car path. The cost functional evaluating the quality of the path $\{\mathfrak{x}(\mathfrak{t},t)\}$ can be written as

$$\mathcal{L}\{\mathfrak{x}\}=\int_t^\infty \exp\left[-\frac{\mathfrak{x}(\mathfrak{t})-x}{\lambda}\right]\mathcal{F}^d(\mathfrak{h},\mathfrak{u},\mathfrak{a})\,d\mathfrak{t}, \tag{3.3a}$$

or

$$\mathcal{L}\{\mathfrak{h}\}=\int_t^\infty \exp\left[-\frac{V}{\lambda}(\mathfrak{t}-t)\right]\mathcal{F}^d(\mathfrak{h},\mathfrak{u},\mathfrak{a})\,d\mathfrak{t}. \tag{3.3b}$$

Forms (3.3a) and (3.3b) correspond to the driver prediction with spatial or temporal limitations, respectively. The cost functional of form (3.3a) was used in Ref. [33]. Subsequently, functional (3.3b) is used because its form simplifies the mathematical manipulations when the velocity V of the lead car is fixed. In particular, dealing with functional (3.3b) we can specify directly the set of trial paths $\mathfrak{S}(t,h,v)$ using solely the time dependence of $\{\mathfrak{h}(\mathfrak{t},t)\}$ of the headway distance. The path variables $\mathfrak{u}(\mathfrak{t},t)$, $\mathfrak{a}(\mathfrak{t},t)$ are completely determined by the dependence $\{\mathfrak{h}(\mathfrak{t},t)\}$:

$$\mathfrak{u}(\mathfrak{t},t)=V-\frac{\partial\mathfrak{h}(\mathfrak{t},t)}{\partial\mathfrak{t}}, \quad \mathfrak{a}(\mathfrak{t},t)=-\frac{\partial^2\mathfrak{h}(\mathfrak{t},t)}{\partial\mathfrak{t}^2}. \tag{3.4}$$

Furthermore, the cost functional (3.3b) matches the driver prediction with temporal limitation, the mechanism which also allows for the interpretation of the recognition region size in terms of $\lambda\sim\vartheta_{\max}\tau_d$. However, both the cost functionals lead practically to the same results for the analyzed situation.

Each one of trial paths originates at time t and starts from (h,v) on the phase plane determined by the headway distance and velocity of the car at the current time t. Since we investigate the car motion inside traffic flow but not processes of leaving it we assume that time variations $h(t)$ of the headway distance are bounded. Therefore the trial paths fulfill

$$\mathfrak{h}(\mathfrak{t},t)\big|_{\mathfrak{t}=t}=h(t), \quad \mathfrak{u}(\mathfrak{t},t)\big|_{\mathfrak{t}=t}=v(t) \tag{3.5}$$

and do exhibit bounded variations only as time goes to infinity. In other words, there is a constant $C>0$ with

$$|\mathfrak{h}(\mathfrak{t},t)|<C \quad \text{for} \quad \mathfrak{t}>t. \tag{3.6}$$

In what follows, the driver behavior will be described by the extremals of the cost functional (3.3b). Using the standard variational technique and taking into account conditions (3.5) and (3.6) the governing equation for these extremals can be derived:

$$\frac{d^2}{d\mathfrak{t}^2}\partial_{\mathfrak{a}}\mathcal{F}^d-2\frac{V}{\lambda}\frac{d}{d\mathfrak{t}}\partial_{\mathfrak{a}}\mathcal{F}^d+\frac{V^2}{\lambda^2}\partial_{\mathfrak{a}}\mathcal{F}^d$$
$$-\frac{d}{d\mathfrak{t}}\partial_{\mathfrak{u}}\mathcal{F}^d+\frac{V}{\lambda}\partial_{\mathfrak{u}}\mathcal{F}^d-\partial_{\mathfrak{h}}\mathcal{F}^d=0. \tag{3.7}$$

To study the spectral properties of Eq. (3.7) it is linearized around the stationary solution (h_V,V). Its eigenfunctions can be found by the *Ansatz*

$$\mathfrak{h}_\zeta(\mathfrak{t})\propto\exp\left(-\zeta\frac{\mathfrak{t}}{\tau}\right).$$

The obtained eigenvalue equation is given by

$$(\zeta+\phi)^2\zeta^2-\Lambda(\zeta+\phi)\zeta+\tfrac{1}{4}\Omega=0, \tag{3.8}$$

with the coefficients

$$\phi=\frac{V}{\vartheta_{\max}}\sigma, \quad \Lambda=\frac{1}{2}\vartheta_{\max}^2\partial_v^2\mathcal{F}>0, \tag{3.9}$$

$$\Omega=2\tau^2\vartheta_{\max}^2\left(\partial_h^2\mathcal{F}-\frac{V}{\lambda}\partial_h\partial_v\mathcal{F}\right) \tag{3.10}$$

and the derivatives are taken at the stationary point (h_V,V). The inequality $\partial_v^2\mathcal{F}>0$ is supposed to hold beforehand, in particular, it is the case for the cost function (2.8). Equation (3.8) possesses four roots, one pair $\{\zeta_+,\zeta_-\}$ of them have positive real parts, the other $\{\zeta'_+,\zeta'_-\}$ have negative ones. Since the eigenfunctions with the eigenvalues $\{\zeta'_+,\zeta'_-\}$ diverge as $\mathfrak{t}\to\infty$ we must omit them by virtue of condition (3.6). Roughly speaking, these divergent eigenfunctions describe the process of a driver leaving traffic flow, for example, to stop the car. Such processes are not under consideration. The former pair of eigenvalues are given by

$$\zeta_\pm=-\frac{1}{2}\phi+\left[\frac{1}{4}\phi^2+\frac{1}{2}\Lambda\pm\frac{1}{2}\sqrt{\Lambda^2-\Omega}\right]^{1/2}. \tag{3.11}$$

So, in a small neighborhood of the stationary point (h_V,V) of Eq. (3.7) any extremal $\mathfrak{h}_{\mathrm{opt}}(\mathfrak{t},t)$ can be written as

$$\mathfrak{h}_{\mathrm{opt}}(\mathfrak{t},t)=h_V+h_+\exp\left(-\zeta_+\frac{\mathfrak{t}-t}{\tau}\right)+h_-\exp\left(-\zeta_-\frac{\mathfrak{t}-t}{\tau}\right), \tag{3.12}$$

where h_+ and h_- are some constants.

Equation (3.7) is of fourth order, so its general solution is specified by four conditions. Solutions that diverge as time goes to infinity have to be omitted (they are related to $\{\zeta'_+,\zeta'_-\}$). Their divergence is due to the exponential cofactor in integral (3.3b) so it is retained beyond a small neighborhood of the stationary point (h_V,V). Therefore, only two conditions are needed to specify the desired solution of Eq. (3.7). In particular, the initial headway distance h and the car

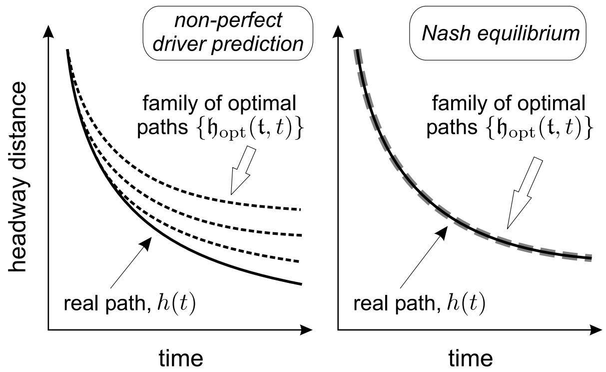

FIG. 3. Illustration of rational-driver strategy. The left panel corresponds to the case where the driver's estimate of the car motion quality is not perfect. This is described by the cost function $\mathcal{F}(\mathfrak{h},\mathfrak{u},\mathfrak{t}|t)$ containing the current time t as its argument. So the found optimal paths $\mathfrak{h}_{\rm opt}(\mathfrak{t},t)$ depend on the time t when the driver plans the further motion. The right panel presents the ideal case where the driver is able to evaluate the motion cost precisely and the corresponding cost function $\mathcal{F}(\mathfrak{h},\mathfrak{u},\mathfrak{t})$ does not contain the time t. So, the optimal path family degenerates into one curve. It should be noted that the latter is correct even if the time $\mathfrak{t}$ enters the cost function via variations of the lead car velocity.

velocity v determine it completely. The latter statement can also be proved for the more general form (3.3a) of the cost functional [33].

B. Rational-driver behavior and Nash equilibrium

In the preceding section the dynamical cost functional was stated. At the next step the driver strategy based on this evaluation of the motion quality should be described (Fig. 3). Following Ref. [33], it is supposed that the driver, first, in planning the further motion chooses the path $\mathfrak{h}_{\rm opt}(\mathfrak{t},t)$ minimizing the motion cost functional (3.3b),

$$\text{driver:} \rightarrowtail \mathfrak{h}_{\rm opt}(\mathfrak{t},t) \Rightarrow \min_{\mathfrak{h}(\mathfrak{t},t)\in\mathfrak{S}(t,h,v)} \mathcal{L}\{\mathfrak{h}(\mathfrak{t},t)\}. \tag{3.13}$$

To do this the driver "solves" Eq. (3.7) subject to conditions (3.5) and (3.6) and, thus, get the optimal path of further driving, $\mathfrak{h}_{\rm opt}(\mathfrak{t},t)$. As noted in the preceding section the optimal path choice is completely determined by the current car velocity v and the headway distance h. The terminal condition, i.e., the goal of reaching the steady-state motion, is implied. Since the driver controls the car motion through choosing the adequate value a of acceleration she has to "find" the second derivative of $\mathfrak{h}_{\rm opt}(\mathfrak{t},t)$ with respect to the former argument $\mathfrak{t}$ [see expression (3.4)] and, then, to "calculate" the result at the current time. Therefore,

$$a(t) = -\lim_{\mathfrak{t}\to t+0} \frac{\partial^2 \mathfrak{h}_{\rm opt}(\mathfrak{t},t)}{\partial \mathfrak{t}^2} \tag{3.14}$$

relates the current car acceleration a to the current headway distance h and the car velocity v.

Formula (3.14) describes the driver's choice at the current moment t of time. To convert it into the governing equation of car motion we adopt the second assumption that the driver performs this choice continuously:

$$a = \mathcal{R}(h,v,V). \tag{3.15}$$

All the car-following models based on this equation, but with different particular forms of the cost function $\mathcal{F}^d(\mathfrak{h},\mathfrak{u},\mathfrak{a})$, may be categorized as the rational car dynamics approximation.

Equation (3.15) holds even if the cost function $\mathcal{F}(\mathfrak{h},\mathfrak{u},\mathfrak{t}|t)$ depends explicitly on the time t at which the driver plans her further motion. Then, the driver evaluation of traffic flow state will change in time. Therefore, the resulting path $\{h(t)\}$ of the real car motion envelops the family of the optimal paths $\{\mathfrak{h}_{\rm opt}(\mathfrak{t},t)\}$ generated at different moments t of time (Fig. 3, left panel).

The present paper assumes driving to be perfect. The driver's choice is based on the precise knowledge of the cost function $\mathcal{F}^d(h,v,a,t)$ depending solely on the current headway distance h, car velocity v, acceleration a, and, may be, the current time t. It does not depend on the moment of time when the driver evaluates the possible paths of her further motion. This assumption leads to what is called the Nash equilibrium in game theory [44]. If, at time t_0 the driver has found the optimal path $\mathfrak{h}_{\rm opt}(\mathfrak{t},t_0)$ of the car motion [which meets Eq. (3.7) and does not contain explicitly t_0], then any recomputation a certain time $\mathfrak{t}>t_0$ later gives the same result, $\mathfrak{h}_{\rm opt}(\mathfrak{t},t)=\mathfrak{h}_{\rm opt}(\mathfrak{t},t_0)$ for $\mathfrak{t}\geqslant t$. In other words, if at time t_0 the driver has chosen an optimal path $\mathfrak{h}_{\rm opt}(\mathfrak{t},t_0)$ then the further motion will be described by it independently of whether the driver follows it without correction or optimizes the car motion continuously (Fig. 3, right panel).

The Nash equilibrium in the driver strategy enables us to conclude that Eq. (3.7) describes the real car dynamics, not only the imaginary paths existing in the driver's mind during her planning of the further motion. In particular, it can be rewritten in a form containing the real acceleration a, velocity v, and headway distance h:

$$\left(\frac{d^2a}{dt^2} - 2\frac{V}{\lambda}\frac{da}{dt} + \frac{V^2}{\lambda^2}a\right) - \frac{\vartheta^2_{\max}}{2\tau^2}\left(\frac{d}{dt}\partial_v\mathcal{F} - \frac{V}{\lambda}\partial_v\mathcal{F} + \partial_h\mathcal{F}\right) = 0. \tag{3.16}$$

In the vicinity of the stationary point (h_V,V) its solution is actually given by *Ansatz* (3.12), which immediately leads us to the following expressions for the amplitudes:

$$h_+ = \frac{\tau(v-V) - \zeta_-(h-h_V)}{\zeta_+ - \zeta_-}, \tag{3.17a}$$

$$h_- = \frac{\zeta_+(h-h_V) - \tau(v-V)}{\zeta_+ - \zeta_-} \tag{3.17b}$$

and relates the car acceleration a to the car velocity v and the headway distance h,

$$a = -\frac{(\zeta_+ + \zeta_-)}{\tau}\left[(v-V) - \frac{\zeta_-\zeta_+}{(\zeta_+ + \zeta_-)}\frac{(h-h_V)}{\tau}\right]. \tag{3.18}$$

These results are analyzed individually.

1. Optimal driving condition

The stationary point $(h=h_V, v=V, a=0)$ of Eq. (3.16) gives the headway h_V which the driver chooses in order to follow the lead car at speed V. In particular, the expression

$$\partial_h \mathcal{F} - \frac{v}{\lambda}\partial_v \mathcal{F} = 0 \tag{3.19}$$

specifies the relationship between the values of the headway h and the car velocity v for the stationary traffic flow imitated by the given car-following problem. Solving Eq. (3.19) for the car velocity v the optimal velocity approximation is obtained, $v=\vartheta_{\rm opt}(h)$. In particular, for the specific form (2.8) of the cost function $\mathcal{F}(h,v)$ we immediately get for $l \ll h \ll \lambda$

$$\vartheta_{\rm opt}(h) = \vartheta_{\max}\frac{h^2}{h^2+D^2}, \tag{3.20}$$

where the spatial scale D is given by expression

$$D = \sqrt{\frac{\lambda l}{2}}, \quad l \ll D \ll \lambda. \tag{3.21}$$

Relation (3.20) or similar sigmoid functions are widely used in the current literature. In particular, for $l=1$ m and $\lambda=300$ m expression (3.21) gives the estimate $D\approx 12$ m typically ascribed to the spatial scale D.

2. Linear governing equation for the car-following problem

By introducing the time scale $\tau_v=\tau/(\zeta_+ + \zeta_-)$ and the coefficient

$$g_h = \frac{\zeta_-\zeta_+}{(\zeta_+ + \zeta_-)^2}, \tag{3.22}$$

the car dynamics equation (3.18) can be rewritten as

$$a = -\frac{1}{\tau_v}\left[(v-V) - g_h\frac{(h-h_V)}{\tau_v}\right]. \tag{3.23}$$

Equation (3.23) plays a significant role in models that describe non-rational-driver behavior [45], where expression (3.23) specifies an optimal acceleration that could be chosen by rational drivers and the parameter g_h was introduced phenomenologically. Here, g_h can be computed using the cost function (2.8). It is plotted as well as the ratio τ_v/τ as a function of the variable Ω (Fig. 4). This representation is due to the fact that the coefficient $\Lambda \sim 1$ [for the cost function (2.8) $\Lambda \simeq 1$ for $l \ll h \ll \lambda$], the coefficient $\phi \ll 1$ by virtue of the adopted assumption (3.2), whereas the coefficient Ω varies around unity. Below, the latter will be justified and the dependence of the coefficient Ω on the car motion state will be analyzed. As seen in Fig. 4 the time scales τ_v and τ practically coincide with each other and the parameter g_h is a small value.

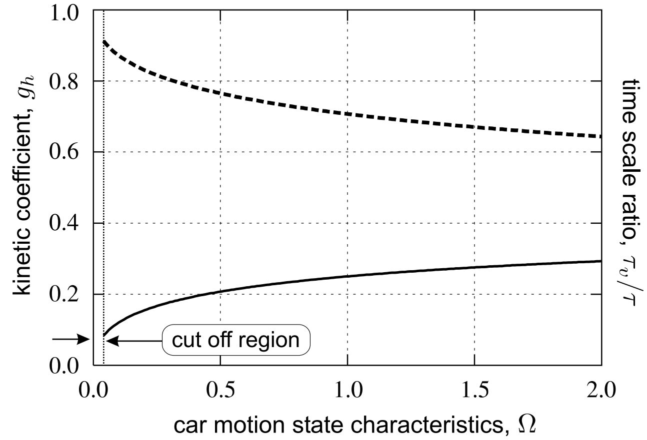

FIG. 4. The kinetic coefficient g_h and the ratio τ_v/τ vs Ω. In drawing these curves, expression (3.11) with $\Lambda=1$ and $\phi=0$ was used. The solid curve is $g_h(\Omega)$, the dashed curve is $\tau_v/\tau(\Omega)$.

3. Following-the-leader model versus optimal velocity model

As discussed already there are two stimuli for a driver to change the current motion state. One is the velocity difference $v-V$ between the leading car and her car. The second is the difference between the current speed v and the optimal speed $\vartheta_{\rm opt}(h)$ as function of headway. Models such as Eq. (1.4) take into account both of them, leaving the determination of the weight coefficients to appropriate empirical or experimental data. The results obtained so far actually allow to predict these coefficients.

This can be demonstrated with the combined model (1.4a). Linearizing Eq. (1.4a) around the stationary point (h_V, V) and comparing the result with Eq. (3.18) we get

$$\kappa = \frac{\zeta_-\zeta_+}{(\zeta_+ + \zeta_-)}\left[\frac{\tau d\vartheta_{\rm opt}}{dh}\bigg|_{h=h_V}\right]^{-1}. \tag{3.24}$$

This expression gives κ as function of the car motion state. Figure 5 plots this dependence for the cost function (2.8). As seen in Fig. 5 the weight coefficient κ is small for all interesting values of the headway distance in the car-following

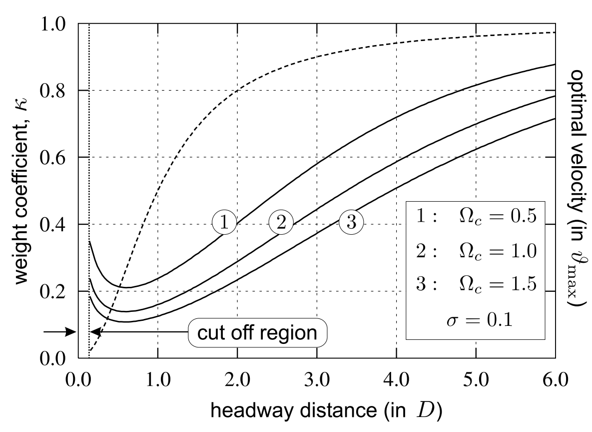

FIG. 5. The weight coefficient κ of the combined model (1.4a) vs the stationary headway h_V. For comparison, the dashed curve shows the optimal velocity, Eq. (3.20).

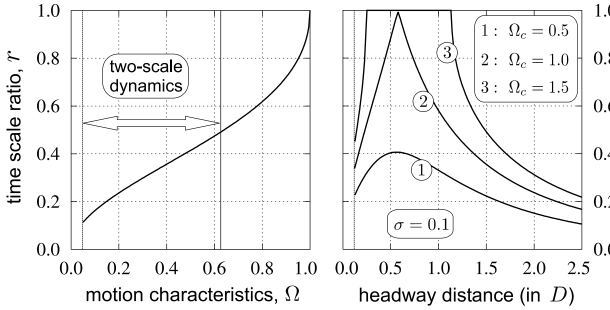

FIG. 6. The ratio of time scales $r=|\zeta_-|/|\zeta_+|$ of the corresponding eigenfunctions describing the car motion relaxation as a function of Ω or the headway h_V when Ω_c is fixed. Computations in this figure used cost function (2.8) and $\phi=0$.

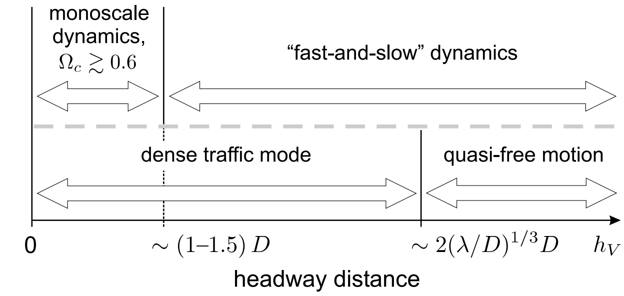

FIG. 7. Possible types of the car motion state classified using different properties of the car dynamics and their mutual arrangement depending on h_V.

regime. Only for small headways $h_V \sim l$ (dense jams) or for large headways h_V exceeding D substantially (free flow) κ approaches unity.

4. Different types of the car dynamics

As the headway distance h_V varies in the interval $l \ll h_V \ll \lambda$ the coefficient $\Omega(h_V)$ changes essentially, leading to qualitatively different car dynamics. For example, within the parabolic approximation for the cost function

$$\Omega = 4\sigma \frac{\tau d\vartheta_{\rm opt}(h)}{dh}\bigg|_{h=h_V} = 8\sigma^2 \frac{\lambda D^2 h_V}{(h_V^2+D^2)^2}, \qquad (3.25)$$

attaining the maximum

$$\Omega_{\max} = \frac{3\sqrt{3}(\tau\vartheta_{\max})^2}{2D\lambda} = \frac{3\sqrt{3}\sigma^2}{2}\frac{\lambda}{D} \qquad (3.26)$$

at $h_\Omega = D/\sqrt{3}$. For $\tau \sim 1$ s, $\vartheta_{\max} \sim 100$ km/h, $D \sim 15$ m, and $\lambda \sim 300$ m, respectively, $\Omega_{\max} \sim 0.5$ is obtained. When $\Omega(h_V) > 1$ the relaxation exhibits damped oscillations around the stationary point (h_V, V). This is due to the eigenvalues $\{\zeta_+, \zeta_-\}$ [see expression (3.11)] having nonzero imaginary parts. Since for $\Omega(h_V) \gtrsim 1$ their real and imaginary parts are of the same order, the car motion relaxation is characterized by one time scale about one (in units of τ). This is true also for $\Omega(h_V) \lesssim 1$ because $\zeta_+ \approx \zeta_-$ there, although the dynamics is a pure fading process now. For still smaller values of $\Omega(h_V) \approx 0.6$ the ratio $r=|\zeta_-|/|\zeta_+|$ of these eigenvalues becomes smaller than one-half, which may be defined as the two-scale regime of the dynamics, see Fig. 6.

The change between the two-scale and the one-scale regime corresponds to a value of $h_V \approx (1\text{–}1.5)D$. Below this value, the dynamical behavior is one-scale, above it is two-scale and will be called "fast-and-slow" in the following.

Concerning with the fast-and-slow dynamics the velocity relaxation and the headway relaxation can be analyzed individually. According to expressions (3.17) the initial difference $h-h_V$ contributes mainly to the eigenfunction with the eigenvalue ζ_-. Thereby, the velocity difference $v-V$ contributes mostly to the amplitude h_+. The amplitude h_- also contains the term of the same magnitude, however, the time scale τ/ζ_- on which the corresponding eigenfunction varies is much bigger than the time scale τ/ζ_+ of the other eigenfunction. So, the velocity relaxation falls on the first eigenfunction. Thus, the velocity difference $v-V$ disappears practically completely during the time τ/ζ_+, which forms the "fast" stage of the car relaxation. At the next "slow" stage of duration τ/ζ_+ the headway deviation from the equilibrium value h_V disappears. To summarize, the fast-and-slow car dynamics is a two-stage process where the velocity difference between the cars is eliminated first. Later on, the headway is optimized. While this is being done, the resulting velocity difference is not essential and the driver can govern the car motion without the necessity of responding fast.

Other important characteristics, separating dense traffic and quasifree flow, can be derived as follows. The solution of the eigenvalue equation (3.8) depends actually on two values ϕ and Ω, since $\Lambda \simeq 1$ for the cost functional (2.8). The parameter $\phi < \sigma \ll 1$ by virtue of the adopted assumption (3.2). The maximum of Ω attained at $h_V = D/\sqrt{3}$ is much larger than σ^2 as it results from expression (3.26). However, as the headway distance increases the value $\Omega(h_V)$ decreases as h_V^{-3} [see expression (3.25)] whereas $\phi \to \sigma$. So there is a value h_c of the headway distance at which both these terms contribute to the eigenvalues to the same extent. To find h_c consider $h \gg D$ where $\Omega(h_V) \ll 1$ and $\phi \approx \sigma$, thereby, $\zeta_+ \simeq 1$ and

$$\zeta_- \simeq \frac{\Omega(h_V)}{2[\sigma + \sqrt{\sigma^2 + \Omega(h_V)}]} \qquad (3.27)$$

by virtue of formula (3.11). So, h_c is specified by

$$\Omega(h_c) = \sigma^2 \Rightarrow h_c = 2\left(\frac{\lambda}{D}\right)^{1/3} D. \qquad (3.28)$$

The value h_c divides the headway distance region into two parts, see Fig. 7. When $h_V \gtrsim h_c$ the velocity of the following car is close to $\vartheta_{\max}$, so this type of car motion will be referred to as the quasifree flow. In this case the eigenvalue equation (3.8) cannot be simplified, so to describe the quasifree flow the cost functional (3.3b) only in its full form may be used. In the opposite case, $h \ll h_c$, which will be called dense traffic mode, the term ϕ is ignorable, reducing the eigenvalue equation (3.8) to

$$\zeta^4-\Lambda\zeta^2+\tfrac{1}{4}\Omega=0. \tag{3.29}$$

Moreover, in the latter case the description of car dynamics can be reduced to the standard form of classical mechanics, enabling us to analyze the nonlinear stage of the car dynamics.

IV. NONLINEAR CAR DYNAMICS

A. Effective cost functional for dense traffic flow

In the dense traffic limit $h\ll h_c$, the variational principle based on optimizing the cost functional (3.3b) can be simplified essentially. In the given limit the characteristic time scales of the car dynamics are τ and $\tau/\sqrt{\Omega(h_V)}$, with both of them being small in comparison with $\lambda/\vartheta_{\max}$. The latter follows from the adopted assumption (3.2) and the condition $\Omega(h_V)\gg\sigma^2$ for $h\ll h_c$.

In this case, as shown in the Appendix, the cost functional (3.3b) can be replaced by the following effective functional whose integrand does not contain a time dependent factor:

$$\mathcal{L}\{h(t)\}=\int_t^\infty \mathcal{F}_{\rm eff}(h,v,a|V)dt', \tag{4.1}$$

where the integrand, which will be called the Lagrangian of the car dynamics, is given by

$$\mathcal{F}_{\rm eff}(h,v,a|V)=\mathcal{F}^d(h,v,a)-\left(\frac{V}{\lambda}h+v\right)\partial_v\mathcal{F}|_{V,h_V}. \tag{4.2}$$

Interestingly, $\mathcal{F}_{\rm eff}(h,v,a|V)$ contains the lead car velocity V as a parameter and attains its extremal value with respect to h at the point h_V corresponding to the optimal driving with the velocity V. The term $v\,\partial_v\mathcal{F}|_{V,h_V}$ has been introduced for the sake of convenience only; it does not affect the extremal equation but enables the Lagrangian to attain a minimum with respect to the car velocity v at the stationary point (h_V,V).

Both functionals (3.3b) and (4.1) possess the same extremals to the first order in the small parameter $\sigma/\sqrt{\Omega(h_V)}$. In particular, the extremals of functional (4.1) meet the equation

$$\frac{2\tau^2}{\vartheta_{\max}^2}\frac{d^2a}{dt^2}-\frac{d}{dt}\partial_v\mathcal{F}-\partial_h\left(\mathcal{F}-\frac{V}{\lambda}\partial_v\mathcal{F}|_{V,h_V}h\right)=0. \tag{4.3}$$

It corresponds directly to the initial full Eq. (3.16) where the second and third terms in the former parentheses are ignored whereas the second term in the latter parentheses is replaced by its value taken at the stationary point. It is justified because these terms are due to variations of the time dependent cofactor.

Keeping in mind its following applications we rewrite Lagrangian $\mathcal{F}_{\rm eff}(h,v,a|V)$ also in the form

$$\mathcal{F}_{\rm eff}(h,v,a|V)=\frac{\tau^2a^2}{\vartheta_{\max}^2}+\mathcal{F}_0(v)+\mathcal{F}_{\rm int}(h,v|V). \tag{4.4}$$

In particular, for the cost function (2.8) we get

$$\mathcal{F}_0(v|V)=\frac{(v-V)^2}{\vartheta_{\max}^2}+\left(1-\frac{V^2}{\vartheta_{\max}^2}\right), \tag{4.5}$$

$$\mathcal{F}_{\rm int}(h,v|V)=\frac{\Omega h_V^2}{4\vartheta_{\max}^2\tau^2}\left(\frac{v^2}{V^2}\frac{h_V}{h}+\frac{h}{h_V}\right). \tag{4.6}$$

In obtaining formula (4.6) we have taken into account expression (3.20) relating the velocity V to the headway distance h_V, expression (3.25), and omitted some insignificant terms. The latter term in expression (4.5) can be also omitted because it has no effect on the extremal equation.

The most essential feature of the effective cost functional (4.1) is the absence of a time dependent cofactor. This enables us to reformulate the car dynamics in terms of autonomous Hamiltonian equations.

B. Hamiltonian description of car dynamics

Finding the extremals of functional (4.1) can be done as follows. The phase space $\{h,v,a\}$ is expanded to $\{h,\dot h,v,\dot v\}$ by adding the relationship between the headway distance h and the car velocity v as an additional constraint: Minimize

$$\int_0^\infty \mathcal{F}_{\rm eff}(h,v,\dot v)dt \tag{4.7}$$

subject to the dynamical equation

$$\dot h=V-v \tag{4.8}$$

and the initial and final conditions

$$h(0)=h_0,\quad v(0)=v_0, \tag{4.9}$$

$$h(\infty)=h_V,\quad v(\infty)=V. \tag{4.10}$$

For the sake of simplicity the parameter V has been omitted in the list of variables of $\mathcal{F}_{\rm eff}$ and the current time is set to zero. By using a Lagrange multiplier, problem (4.7) is rewritten in the standard form; namely, the extremals of

$$\int_0^\infty[\mathcal{F}_{\rm eff}(h,v,\dot v)+p(\dot h+v-V)]dt \tag{4.11}$$

are sought. They are defined on the extended set of variables $\{h(t),v(t),p(t)\}$ and subject to conditions (4.9) and (4.10). The extremals of functional (4.11) obey the classical Lagrange equations.

In constructing the Hamiltonian $\mathcal{H}(h,v,p,q)$ that produces the same extremal equations the Pontryagin technique is used [46]. Introducing the new variable q,

$$q=\frac{\partial\mathcal{F}_{\rm eff}(h,v,\dot v)}{\partial\dot v}, \tag{4.12}$$

and solving for $\dot v$, i.e., finding $\dot v=\dot v(h,v,q)$, the desired Hamiltonian is written as

$$\mathcal{H}(h,v,p,q)=q\dot{v}-\mathcal{F}_{\rm eff}(h,v,\dot{v})-p(v-V). \quad (4.13)$$

It can be demonstrated directly that the desired extremals meet the standard Hamiltonian system of equations

$$\dot{h}=\partial_p\mathcal{H}, \quad \dot{v}=\partial_q\mathcal{H}, \quad (4.14)$$

$$\dot{p}=-\partial_h\mathcal{H}, \quad \dot{q}=-\partial_v\mathcal{H}. \quad (4.15)$$

Again, the optimization of functional (4.7) actually leads to a boundary value problem because the extremal is specified by both the initial and final conditions (4.9) and (4.10). The Hamiltonian approach [Eqs. (4.14) and (4.15)] also shares this property; namely, the initial values of the variables h and v are given by conditions (4.9), whereas the initial values p_0 and q_0 of the quasimomenta p and q should be chosen such that the system tends to the stationary point (h_V,V) as $t\rightarrow\infty$. It is the same situation that we met in dealing with Eq. (3.16).

However, the Hamiltonian description has a certain advantage. First, the Hamiltonian itself is conserved,

$$\frac{d\mathcal{H}(h,v,p,q)}{dt}=0. \quad (4.16)$$

Second, an additional autonomous first integral of the system (4.14),(4.15), can be found, at least, for a certain stage of the car dynamics. Fortunately, this is the case for the fast-and-slow car dynamics analyzed below.

Note, that this Hamiltonian description of the car motion relaxation towards the stationary state does not contradict the conservation of phase volume being the general property of Hamiltonian systems. The matter is that the relaxation process is described by only *one* path leading to the stationary point of the saddle type. Other possible paths are not considered.

Physical meaning of the Hamiltonian variables

The two Eqs. (4.14) can be understood readily. The equation for $\dot{h}$ is just $\dot{h}=V-v$, while the equation for $\dot{q}$, or, what is the same, expression (4.12) shows that the quasimomentum q is proportional to the car acceleration $a=\dot{v}$:

$$q=\frac{2\tau^2}{\vartheta_{\rm eff}^2}a. \quad (4.17)$$

The physical meaning of the quasimomentum p is more complex. To clarify it, rewrite the equation line (4.15) in terms of partial derivatives of the Lagrangian $\mathcal{F}_{\rm eff}(h,v,\dot{v})$; namely, the definitions of the quasimomentum q (4.12) and the Hamiltonian (4.13) enables us to represent these equations as

$$\dot{q}=\partial_v\mathcal{F}_{\rm eff}+p, \quad (4.18)$$

$$\dot{p}=\partial_h\mathcal{F}_{\rm eff}. \quad (4.19)$$

In particular, expressions (4.17)–(4.19) immediately lead, as it must, to Eq. (4.3).

Expression (4.18) demonstrates that the quasimomentum p includes the rate $\dot{a}$ of acceleration variations. In this way the variable $\dot{a}$ implicitly enters the Hamiltonian $\mathcal{H}(h,v,p,q)$ specifying a hypersurface in the four-dimensional phase space $\{h,v,a,\dot{a}\}$. Even more, by ignoring the dependence of $\mathcal{F}_{\rm eff}(h,v,\dot{v})$ on the headway distance h, the velocity relaxation towards the stationary value V is governed by the equation

$$\dot{q}=\partial_v\mathcal{F}_{\rm eff}.$$

It results from the effective cost functional (4.1) where the car velocity v is treated as a primary argument. As will be seen below, there is a stage of the car dynamics where this assumption is justified and the quasimomentum $p=0$ is conserved. Only the dependence of the Lagrangian $\mathcal{F}_{\rm eff}(h,v,\dot{v})$ on the headway distance h, see Eq. (4.19), causes variations in the quasimomentum p. Since the velocity control is of primary importance in the driving strategy, the variable p can be regarded as a certain measure of the necessity to control also the headway distance when eliminating the velocity difference.

C. Fast-and-slow dynamics

For $\Omega\ll 1$ the last term on the right-hand side of expression (4.4) is small. Since the headway h enters the Hamiltonian (4.13) exactly via this term, time variations in the quasimomentum p are retarded by virtue of the first equation at line (4.15). The same follows from Eq. (4.19). The other variables h,v,q can vary substantially on scales independent of the value $\Omega\ll 1$. Then, the car dynamics can exhibit multiscale relaxation. In particular, Sec. III B 4 demonstrated this fact; namely, it has been shown that in the given limit the system dynamics comprises the fast and slow stages. Exactly the time scale of the latter stages depends on the value $\Omega\ll 1$. During the former stage the quasimomentum p should be practically constant, i.e., it is approximately a first integral of the system (4.14),(4.15). This property, together with the conservation of Hamiltonian (4.13), allows the integration of the system of equations (4.14) and (4.15). During the following slow stage a quasistationary approximation can be used to analyze the system evolution.

The results below are exemplified with the cost function (2.8). They can be easily generalized to other cost functions.

1. Fast stage

At the zeroth approximation in the small parameter Ω the quasimomentum p is a constant. During the fast stage mainly the velocity difference $v-V$ is eliminated, so Eq. (4.18) formally describes the relaxation process of the car velocity v to the stationary value V. Therefore, by virtue of expression (4.5), the quasimomentum p takes zero value, $p=0$. In this limit the Lagrangian component $\mathcal{F}_{\rm int}(h,v|V)$ as well as the additive constant of the component $\mathcal{F}_0(v|V)$ can be omitted. So, the Hamiltonian (4.13) becomes

$$\mathcal{H}(v,a)=\mathcal{H}_a(a)-\mathcal{H}_v(v),$$

where

$$\mathcal{H}_a(a)=\frac{\tau^2 a^2}{\vartheta_{\max}^2}, \quad \mathcal{H}_v(v)=\frac{(v-V)^2}{\vartheta_{\max}^2}, \tag{4.20}$$

and the noncanonical variables $\{h,v,a\}$ are used.

Since the Hamiltonian $\mathcal{H}(v,a)$ is conserved and the fast stage describes the velocity relaxation to V the car dynamics obeys the equation

$$\mathcal{H}_a(a)=\mathcal{H}_v(v). \tag{4.21}$$

This immediately gives the relationship between the acceleration a and the velocity v:

$$a=-\frac{1}{\tau}(v-V). \tag{4.22}$$

The sign in the latter equation has been chosen so to allow for the system relaxation.

Equation (4.21) is, in fact, of the general form and the linear form of Eq. (4.22) is due to the adopted quadratic *Ansatz* of the cost function but not a consequence of linearization. Besides, equality (4.21) can be read as follows. The comparison of the Hamiltonian parts $\mathcal{H}_a(a)$ and $\mathcal{H}_v(v)$ with the corresponding components of Lagrangian (4.4) demonstrates that the function $\mathcal{H}_a(a)$ actually measures the cost of the car acceleration and the function $\mathcal{H}_v(v)$ does the same with respect to the car motion relative to the optimal driving conditions. Thereby, during the fast stage the driver corrects the car dynamics so that the cost of acceleration be equal to the cost of current motion measured relative to the stationary conditions.

The headway distance h does not enter explicitly the governing equation of the fast stage. However, it varies during the fast stage and finally attains a value h_* differing from the initial value h_0. In the adopted simple approximation of the cost function, Eq. (4.22) enables us to estimate easily the value h_*; namely, the direct integration of Eq. (4.22) yields the relationship

$$h_*=h_0-\tau(v_0-V). \tag{4.23}$$

Applying to formula (4.33) it can be seen that expression (4.23) holds until the headway distance h becomes too small and it is impossible to ignore the effect of the term $\mathcal{F}_{\text{int}}(h,v)$, i.e., when

$$h\lesssim\sqrt[3]{\Omega}h_V\ll h_V. \tag{4.24}$$

The car dynamics in the region of small values of the headway distance when the probability of collision is high is worthy of an individual consideration. Here we only touch on this problem by assuming the collision to happen when the value h_* given by expression (4.23) becomes equal to zero. This assumption is justified as a rough approximation due to estimate (4.24), see Fig. 8. The boundary of the collisionless region can be shifted to the right substantially by dropping the assumed quadratic dependence of the cost function on acceleration. This will be done elsewhere.

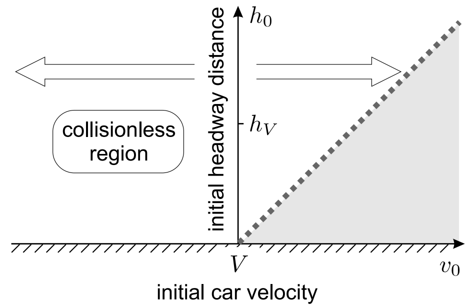

FIG. 8. Phase region of the initial values of the headway distance h_0 and the car velocity v_0 where the car dynamics is collision-free [based on Eq. (4.23)].

2. Slow stage

When the system attains a quasiequilibrium with respect to the car velocity v its further evolution is due to the direct dependence of the Hamiltonian $\mathcal{H}(h,v,q,p)$ on the headway distance h. It enters the Hamiltonian via $\mathcal{F}_{\text{int}}(h,v|V)$. As a result, the velocity difference $v-V$ and the acceleration a should be small. Keeping this in mind, Eq. (4.18) can be solved for p. Substitution of the obtained expression into Eq. (4.19), and the further linearization with respect to the variable $v-V$ and the component $\mathcal{F}_{\text{int}}$, results in

$$\ddot{q}-a\,\partial_v^2\mathcal{F}_0|_{v=V}=\partial_h\mathcal{F}_{\text{int}}. \tag{4.25}$$

If τ_s is the characteristic time scale of the slow stage then

$$(v-V)\sim\frac{h-h_V}{\tau_s}, \quad a\sim\frac{h-h_V}{\tau_s^2}, \quad q\propto\dot{a}\sim\frac{h-h_V}{\tau_s^3}.$$

These estimates together with Eq. (4.25) lead to

$$\tau_s\sim\Omega^{-1/2}\tau. \tag{4.26}$$

In particular, the rate $\dot{q}$ of time variations in the quasimomentum q scales as $\dot{q}\propto\Omega^2$ for $\Omega\to 0$. Since the Lagrangian $\mathcal{F}_{\text{eff}}$ has a minimum at $h=h_V$ the quasimomentum p can be estimated as

$$p\sim\tau_s\partial_h\mathcal{F}_{\text{int}}\propto\Omega^{1/2}(h-h_V)$$

by virtue of Eq. (4.19). In the limit $\Omega\ll 1$ the term $\dot{q}$ can be ignored in comparison with the quasimomentum p. Then, Eq. (4.18) immediately leads to the relation

$$p=-\partial_v\mathcal{F}_{\text{eff}}, \tag{4.27}$$

which is no more than the standard expression for the momentum of a system described by the Lagrangian $\mathcal{F}_{\text{eff}}(h,v,0)$ with $v=V-\dot{h}$.

By the same reasons all the terms in Hamiltonian (4.13) containing the quasimomentum q may be omitted. Then the substitution of Eq. (4.27) into Eq. (4.13) yields the Hamiltonian of the slow stage

$$\mathcal{H}(h,v)=\mathcal{H}_v(v)-\mathcal{H}_h(h), \tag{4.28}$$

where

$$\mathcal{H}_h(h)=\mathcal{F}_{\rm int}(h,V|V)-\mathcal{F}_{\rm int}(h_V,V|V)=\frac{\Omega\, h_V^2}{4\vartheta_{\max}^2\tau^2}\frac{(h-h_V)^2}{h_V h}. \tag{4.29}$$

In the Lagrangian component $\mathcal{F}_{\rm int}(h,v|V)$ the car velocity v has been replaced by its stationary value V, and some negligible terms have been omitted. Then the conservation of the Hamiltonian $\mathcal{H}(h,v)$ during the system dynamics gets the form

$$\mathcal{H}_v(v)=\mathcal{H}_h(h), \tag{4.30}$$

which leads to the following governing equation:

$$v-V=\frac{\sqrt{\Omega}}{2\tau}(h-h_V)\sqrt{\frac{h_V}{h}}. \tag{4.31}$$

The conservation law (4.30) can be understood in analogy to that for the fast stage. However, during the slow stage of the car dynamics the car velocity plays the role of the control parameter. The driver changes the speed to correct the headway distance. Again the cost of deviation of the car velocity from the stationary value is chosen to be equal to the cost of the car motion measured relative to the optimal conditions.

It should be pointed out that the slow stage of the car dynamics is governed by the conservation law, expression (4.30), which contains only the headway distance h and the velocity v. At the first approximation, the acceleration a does not enter at all. In this meaning, the slow stage is similar to other physical systems. However, the stationary point of the car dynamics is a saddle point rather than a minimum.

Up to now, the two different stages of car-following discussed already in Sec. III B 4 have been identified with two different conservation laws. Both of them can be unified into just another effective conservation law interpolating expressions (4.21) and (4.30):

$$\mathcal{H}_v(v)=\mathcal{H}_a(a)+\mathcal{H}_h(h). \tag{4.32}$$

Within the same accuracy it is possible to interpolate directly Eqs. (4.22) and (4.31), leading to

$$a=-\frac{1}{\tau}\left[(v-V)-\frac{\sqrt{\Omega}}{2\tau}\left(\frac{h_V}{h}\right)^{1/2}(h-h_V)\right]. \tag{4.33}$$

Besides, the proposed interpretations of the conservation laws (4.21) and (4.30) enable us to formulate a generalized principle of adequate control. It declares that the effective cost of correcting the car motion via changing the car acceleration (fast stage) or the car velocity (slow stage) is equal to the cost of the current state of car motion measured with respect to the optimal driving conditions.

V. CONCLUSION AND DISCUSSION

A variational approach to the description of car dynamics has been developed in case of a car following a lead car moving at a constant speed. To derive governing equations for the following car motion the driver preference has been used to construct a cost functional. Its extremals specify the optimal paths of the further motion for the following car. Applying to the general properties of the driver behavior we analyzed the basic properties of the cost function and proposed a simple parabolic *Ansatz*, which, nevertheless, catches typical features of traffic properties.

The concept of a rational driver is formulated. It comprises the assumptions that the driver follows the optimal paths and corrects the car motion continuously. In this case the optimal path is a Nash equilibrium of the system, which is defined as follows. If the driver has chosen an optimal path at a certain moment of time then no further correction of the car motion is necessary because it leads to the same result. As the consequence of the Nash equilibrium the extremals of the cost functional specify the real dynamics of cars with rational drivers although originally they determine only the imaginary paths in the driver's mind when planning the further motion. In this way we obtained several results. The optimal velocity approximation has been derived. The weight coefficient entering a car-following model combining the following-the-leader model and the optimal velocity model has been found depending on the headway distance. As an important result it has been shown that the car dynamics can be categorized under different types according to its properties. First, we have shown that there can be two types of the car relaxation towards the stationary motion, the monoscale dynamics and the fast-and-slow dynamics. Second, we singled out the quasifree motion and the dense traffic mode.

The variational technique for the latter mode can be simplified essentially; namely, it is possible to reformulate the cost functional so that it does not contain a time dependent cofactor. As a result, the autonomous Hamiltonian description for the car dynamics has been constructed. For the dense traffic mode the fast-and-slow dynamics has been analyzed for the nonlinear stage. In particular, different conservation laws for these stages have been found. A generalized principle of adequate car motion control has been proposed.

We assumed that the lead car moves at a fixed speed. If it is not the case two different situations should be singled out. For perfect drivers who can predict rigorously the motion of the car ahead the main results still hold. Otherwise the cost function will depend not only on the current time but also, what is crucial, on the time when the driver has started to evaluate the further car dynamics. This disturbs the Nash equilibrium and the car correction should be carried out continuously. Briefly this problem was studied in Ref. [33] but it actually requires a more detailed investigation.

Beyond the rationality

Real drivers have certain limitations. They are not capable of finding the optimal path precisely and they cannot correct the car motion continuously. So, the concept of rational-driver behavior is just the first approximation of the real situation and deviations from this perfect behavior should be analyzed consistently. A first step towards this problem can be found in Ref. [45]. Here we justify some assumptions adopted there and substantiated them in a different way.

Let us again concern the driver evaluation of the car motion quality. For a given path of the car's further motion $\{\mathfrak{h}(\mathfrak{t},t)\}$ the cost functional (3.3b) is written as

$$\mathcal{L}\{\mathfrak{h}\}=\int_t^\infty d\mathfrak{t}\, e^{-(V/\lambda)(\mathfrak{t}-t)}\left[\frac{\tau^2\mathfrak{a}^2}{\vartheta_{\max}^2}+\left(1-\frac{\mathfrak{u}}{\vartheta_{\max}}\right)^2+\frac{\mathfrak{u}^2}{\vartheta_{\max}^2}\frac{l}{\mathfrak{h}}\right]. \tag{5.1}$$

Expression (5.1) can be represented also in the form

$$\mathcal{L}\{\mathfrak{h}\}=\int_t^\infty d\mathfrak{t}\, e^{-(V/\lambda)(\mathfrak{t}-t)}\left[\frac{\tau^2\mathfrak{a}^2}{\vartheta_{\max}^2}+\frac{(\mathfrak{u}-V)^2}{\vartheta_{\max}^2}+\frac{V^2}{\vartheta_{\max}^2}\frac{l\delta^2}{h_V^2(h_V+\delta)}\right]+\frac{2V}{\vartheta_{\max}^2}(h-h_V)+\mathcal{L}_0, \tag{5.2}$$

where $\delta(\mathfrak{t},t)$ describes the deviation of the given path from the stationary car motion trajectory,

$$\mathfrak{h}(\mathfrak{t},t)=h_V+\delta(\mathfrak{t},t).$$

Here, $\mathcal{L}_0$ is the cost of stationary car motion,

$$\mathcal{L}_0=\frac{\lambda}{V}\left(\frac{V}{\vartheta_{\max}}-1\right)^2. \tag{5.3}$$

In addition, when deriving expression (5.1) we have used

$$\frac{l}{h}=\frac{l}{h_V}-\frac{l\delta}{h_V^2}+\frac{l\delta^2}{h_V^2(h_V+\delta)},$$

integrated various fragments of Eq. (5.1) by parts, assuming $l\ll\mathfrak{h}\ll\lambda$, and dropped terms such as $l/\mathfrak{h}$ wherever possible.

When the driver plans her further motion the current values of the headway h and the velocity v are regarded as the initial conditions. So, when choosing the optimal path she can minimize only the first term $\mathcal{L}_c\{\mathfrak{h}\}$ of expression (5.2). This optimization is implemented through the adequate control over the car acceleration a, so, exactly the acceleration plays the role of the control parameter available for the driver actions.

A real driver can only approximately evaluate the quality of motion. Let us describe the threshold in the driver perception of the motion quality by

$$\mathcal{L}_{\rm thr}=\frac{\lambda}{V}\epsilon_c^2, \tag{5.4}$$

where ϵ_c is a small constant, $\epsilon_c\ll 1$, because definitely a driver can recognize the difference between the state of staying and free motion on an empty road.

When the controllable part $\mathcal{L}_c\{\mathfrak{h}\}$ of the cost motion is much smaller than threshold (5.4) the driver actually has no information of how to govern the car motion. In this case it is natural to assume that he will not do anything with respect to the car driving and, so, will fix the car motion at the current state, including the current acceleration a. Thereby the inequality

$$\mathcal{L}_c\{\mathfrak{h}\}\lesssim\mathcal{L}_{\rm thr} \tag{5.5}$$

determines the region in the phase space $\{h,v,a\}$ inside which the driver cannot control the car motion. In evaluating such a driver behavior with expression (5.2) we can formally treat δ, $\mathfrak{u}$, and $\mathfrak{a}$ as constants independent of one another. So setting $\delta=h-h_v$, $\mathfrak{u}=v$, and $\mathfrak{a}=a$ and taking into account the relation between h_V and V we get from condition (5.5) the approximate boundary of this region in the following form:

$$\tau^2a^2+(v-V)^2+\frac{\Omega(h_V)}{4\tau^2}\frac{h_V}{h}(h-h_V)^2\lesssim\epsilon_c^2\vartheta_{\max}^2. \tag{5.6}$$

We recall that the parameter g_h entering Eq. (3.23) coincides with $\sqrt{\Omega}/(2\tau)$. So we reproduced here the expression for the rational driving boundary as has been introduced in paper [45] by a different line of reasonings.

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APPENDIX: LAGRANGIAN REPRESENTATION OF THE COST FUNCTIONAL

The problem of finding the extremals of the functional

$$L\{h(t)\}=\int_0^\infty dt\, w(t)\mathcal{F}(h,v,a) \tag{A1}$$

defined on a set of paths $\{h(t)\}|_{t=0}^\infty$ is considered. Here, first, the integrand $\mathcal{F}(h,v,a)$ is a given function of the headway distance h, the current velocity $v=V-dh/dt$, and the acceleration $a=-d^2h/dt^2$. Second, the weight factor $w(t)>0$ is a function of time t confined within some time interval $(0,\tau_V)$, i.e., $w(0)\sim 1$, $w(t)\ll 1$ for $t\gg\tau_V$, and $w(t)\to 0$ sufficiently fast as $t\to\infty$. So, $w(0)=1$ can be adopted without loss of generality. Third, the trial paths $\{h(t)\}$ meet the initial conditions

$$h(0)=h_0,\quad v(0)=v_0, \tag{A2}$$

and do not diverge as time goes on, i.e., there is a number C so that

$$|h(t)|<C. \tag{A3}$$

The properties of the functional extremals have been studied already. Keeping in mind the results in Sec. III, a special case is analyzed in the following. Let $\{\tau_d\}$ be the characteristic time scale of the velocity relaxation to the stationary value V. In particular, the values of $\{\tau_d\}$ are estimated by the eigenvalues of the corresponding extremal equation linearized near the stationary point. The case $\tau_d \ll \tau_V$ is considered here to show how functional (A1) can be rewritten to eliminate the time dependent factor $w(t)$. In this way the problem of finding the extremals gets the classical form met in theoretical mechanics.

The stationary point $\{h_V, V, a=0\}$ is determined by the properties of both the function $\mathcal{F}(h,v,a)$ and the factor $w(t)$ [see expression (3.19)]. As a result, at the stationary point [see again expression (3.19)]

$$\partial_a \mathcal{F}_{\rm st}=0, \quad \partial_h \mathcal{F}_{\rm st} \propto \frac{1}{\tau_V}. \tag{A4}$$

The derivative $\partial_v \mathcal{F}_{\rm st}$ does not contain factors similar to τ_d/τ_V. So, $w(t)$ cannot be omitted directly. As a first step the function $\mathcal{F}(h,v,a)$ is replaced by the difference $\Delta\mathcal{F}(h,v,a) := \mathcal{F}(h,v,a) - \mathcal{F}(h_V,V,0)$. This does not affect the extremals. The difference $\Delta\mathcal{F}(h,v,a)$ as a function of t is practically confined within a time interval $(0,T_d)$ whose upper boundary $T_d \gtrsim \max\{\tau_d\}$ and, so, $T_d \ll \tau_V$. Therefore, at the next step expression (A1) is rewritten as

$$L\{h(t)\} \cong \int_0^\infty dt \left\{ \Delta\mathcal{F}(h,v,a) - \frac{t}{\tau_V}(v-V)\partial_v \mathcal{F}(h_V,V,0) \right\}, \tag{A5}$$

which is justified to the first order in the small parameter $\max\{\tau_d\}/\tau_V$ by virtue of expressions (A4) and the fact that the second term in expression (A5) plays a substantial role only when the path $h(t)$ tends to the stationary point. Here the symbol $\cong$ means that functional (A5) possesses the same collection of the extremals as the initial functional $L\{h(t)\}$ within the adopted accuracy and the time dependent factor $w(t)$ has been approximated by

$$w(t) \approx 1 - \frac{t}{\tau_V}. \tag{A6}$$

Formula (A6) can be regarded actually as the definition of the time scale $\tau_V := [-dw(0)/dt]^{-1}$. Taking into account the relation $(v-V) = -dh/dt$ and integrating the second term in expression (A5) by parts yields

$$L\{h(t)\} \cong \int_0^\infty dt \left\{ \Delta\mathcal{F}(h,v,a) - \frac{(h-h_V)}{\tau_V}\partial_v \mathcal{F}(h_V,V,0) \right\}. \tag{A7}$$

Omitting the constant components of the integrand which does not affect the extremals the required result follows:

$$L\{h(t)\} \cong \int_0^\infty dt \left\{ \mathcal{F}(h,v,a) - \frac{h}{\tau_V}\partial_v \mathcal{F}(h_V,V,0) \right\}. \tag{A8}$$

Minimization of functional (A8) gives the desired extremals and integral (A8) does not contain a time dependent factor. The function

$$\mathcal{F}_{\rm eff}(h,v,a|V) := \mathcal{F}(h,v,a) - \frac{h}{\tau_V}\partial_v \mathcal{F}(h_V,V,0) \tag{A9}$$

can be regarded as a Lagrangian of the car dynamics.

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